Non-Noether symmetries and conserved quantities of nonconservative dynamical systems

Jing-Li Fu $^{a,c,*}$, Li-Qun Chen $^{b,c}$

$^a$ Department of Physics of Shangqiu Teachers College, Shangqiu 476000, China
$^b$ Department of Mechanics, Shanghai University, Shanghai 200072, China
$^c$ Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China

Received 5 June 2003; received in revised form 17 August 2003; accepted 21 August 2003
Communicated by R. Wu

Abstract

This Letter focuses on studying non-Noether symmetries and conserved quantities of the nonconservative dynamical system. Based on the relationships among motion, nonconservative forces and Lagrangian, we present conservation laws on non-Noether symmetries for nonconservative dynamical systems. A criterion is obtained on which non-Noether symmetry leads to Noether symmetry in nonconservative systems. The Letter also gives connection between the non-Noether symmetries and Lie point symmetries, and further obtains Lie invariants which form a complete set of invariants. Finally, an example is discussed to illustrate these results.

© 2003 Elsevier B.V. All rights reserved.

MSC: 70H45; 70H33

Keywords: Conservation law; Symmetry; Nonconservative system; Infinitesimal transformation

1. Introduction

It is well known from the Noether’s theorem [1] that the constant of motion for dynamical systems can be associated with continuous transformation of coordinates and time which leave the action integral invariant. Transformation that leaves the equation of motion invariant is called symmetry. Lutzky gave non-Noether conserved quantity of Lie point symmetries and Lie velocity-dependent symmetries for Lagrangian systems. They pointed out that a given symmetry, if conserves the action integral, leads to Noether symmetry [2–4]. Cicogna and Gaeta obtained some remarks on the existence and properties of Lie point symmetries in mechanics. However, they did not obtain conserved quantity corresponding to these systems [5]. Without using either Lagrangians or Hamiltonians, Hojman presented a theorem to obtain directly conserved quantities with Lie symmetries [6]. Mei made major progresses in Lie symmetries for constraint mechanical systems [7]. Nevertheless, in their studies only the conserved quantities of Noether symmetrical type are

* Corresponding author.
E-mail addresses: sqfujingli@163.com (J.-L. Fu), lqchen@online.sh.cn (L.-Q. Chen).
considered. In this Letter, we first give a connection among the motion, nonconservative forces and the Lagrangian for a nonconservative system; then we extend non-Noether symmetries of Lagrangian systems to nonconservative mechanical systems. A formula is derived for the non-Noether conserved quantities of nonconservative symmetries. We further present the connection between the non-Noether and the Lie point symmetries, and provide a condition that can obtain Noether symmetries from non-Noether symmetries of nonconservative systems. Finally, we show that the conserved quantities of Lie point symmetries form a complete set of conserved quantities for nonconservative systems.

2. The non-Noether symmetries of nonconservative dynamical systems

The $n$ degree of freedom mechanical system with Lagrangian $L(t, \mathbf{q}, \dot{\mathbf{q}})$, subjected to a nonpotential generalized force $Q_s(t, \mathbf{q}, \dot{\mathbf{q}})$, has the equations of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s(t, \mathbf{q}, \dot{\mathbf{q}}), \quad (s = 1, \ldots, n).$$

The equations of motion can be put in the form

$$\ddot{q}_s = \alpha_s(t, \mathbf{q}, \dot{\mathbf{q}}).$$

Let us introduce the infinitesimal transformations with respect to time and coordinates

$$t^* = t + \epsilon \xi_0(t, \mathbf{q}),$$

$$q_s^* = q_s + \epsilon \xi_s(t, \mathbf{q}),$$

where $\epsilon$ is an infinitesimal parameter, and $\xi_0, \xi_s$ are infinitesimal generators.

The invariance of Eq. (2) under the infinitesimal transformation (3) leads to the satisfaction of the determining equations

$$\ddot{\xi}_s - \dot{\xi}_s \dot{\xi}_0 - 2\alpha_s, \xi_0 = X^{(1)}(\alpha_s).$$

The infinitesimal generators, $\xi_0(t, \mathbf{q})$ and $\xi_s(t, \mathbf{q})$, form a complete set of the symmetries for nonconservative dynamical systems. The operator $X^{(1)}$ is the generator of the first extended group [2], given by

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{\xi}_0 \dot{\xi}_s) \frac{\partial}{\partial \dot{q}_s},$$

and the vector field

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \alpha_s \frac{\partial}{\partial \dot{q}_s}. \tag{6}$$

represents the total time derivatives along the trajectories of Eq. (2). Then for any function $\phi$

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \dot{q}_s \frac{\partial \phi}{\partial q_s} + \alpha_s \frac{\partial \phi}{\partial \dot{q}_s}. \tag{7}$$

If Eq. (2) leaves invariant under the transformation of continuous Lie group (3), the invariance is called the non-Noether symmetry that is represented by a complete set of generators $\xi_0(t, \mathbf{q}), \xi_s(t, \mathbf{q})$ satisfying Eq. (4) of the nonconservative system.

Eq. (4) can be regarded as a criterion of non-Noether symmetries, i.e.,

**Criterion.** The symmetries are called non-Noether symmetries, corresponding the nonconservative dynamical systems (2) if the infinitesimal generators $\xi_0, \xi_s$ satisfy the determining equation (4).

To derive non-Noether conserved quantities, two relations for nonconservative and conservative dynamical systems are needed.

First, the $n$-dimensional nonconservative dynamical system (1) is written in the form

$$\frac{\partial^2 L}{\partial q_k \partial \dot{q}_k} \ddot{q}_k = \frac{\partial L}{\partial q_k} - \frac{\partial^2 L}{\partial q_k \partial t} - \frac{\partial^2 L}{\partial q_k \partial \dot{q}_s} \dot{q}_s + Q'_s. \tag{8}$$

One can easily demonstrate a relation among $\alpha_s, Q'_s$ and $L$ as [1]

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} - \frac{\partial}{\partial q_k} \left( \frac{M_{k2}}{D} Q'_s \right) + \frac{d}{dt} (\ln D) = 0, \tag{9}$$

where

$$D = \det \left[ \frac{\partial^2 L}{\partial \dot{q}_s} \right] \tag{10}$$

and $M_{k2}$ is the cofactor of $\partial^2 L/\partial \dot{q}_k \partial q_s$ in the matrix formed by the second derivatives.

Second, we remark that if $\xi_0, \xi_s$ satisfy Eq. (4), it can be shown that [2]

$$\dot{X}^{(1)}(\phi) = X^{(1)}(\dot{\phi}) + \dot{\xi} \phi, \tag{11}$$

is satisfied for any function $\phi(t, \mathbf{q}, \dot{\mathbf{q}})$. With these results in hand, one can prove
Theorem 1. If the infinitesimal transformation generators $\xi_0$ and $\xi_s$ satisfy Eq. (4) and the function $f = f(t, q, \dot{q})$ satisfies the following equation
\[
\frac{df}{dt} = \frac{\partial}{\partial \dot{q}_k} \left( M_{ks} \frac{Q_s}{D} \right),
\]
then nonconservative system (2) possesses a conserved quantity
\[
\Phi = 2 \left( \frac{\partial \xi_s}{\partial q_s} - \dot{q}_s \frac{\partial \xi_0}{\partial q_s} \right) - N\dot{\xi}_0 + X^{(1)}(\ln D) - X^{(1)}(f).
\]  

Proof. Move the right-hand side of Eq. (4) to the left, and represent it by $\Pi_s$, which has derivative
\[
\frac{\partial \Pi_s}{\partial q_s} = \frac{d}{dt} \left[ 2 \left( \frac{\partial \xi_s}{\partial q_s} - \dot{q}_s \frac{\partial \xi_0}{\partial q_s} \right) - N\dot{\xi}_0 + X^{(1)}(\ln D) \right] - X^{(1)}(\frac{\partial}{\partial \dot{q}_s} \left( M_{ks} \frac{Q_s}{D} \right)),
\]
which allows Eq. (14) to be rewritten in the form
\[
\frac{\partial \Pi_s}{\partial q_s} = \frac{d}{dt} \left[ 2 \left( \frac{\partial \xi_s}{\partial q_s} - \dot{q}_s \frac{\partial \xi_0}{\partial q_s} \right) - N\dot{\xi}_0 + X^{(1)}(\ln D) \right] - X^{(1)}\left( \frac{\partial}{\partial \dot{q}_s} \left( M_{ks} \frac{Q_s}{D} \right) \right).
\]

This is the main result in the present Letter. We should note that in order to get this conserved quantity, it is necessary to postulate that the equation of motion be derived from a Lagrangian, but the action integral invariant is not essential.

Theorem 1 can be regarded as a criterion to obtain non-Noether conserved quantities associated with nonconservative dynamical systems, where Eq. (12) is a restrained condition of the nonconservative force, and Eq. (13) gives non-Noether conserved quantity of the nonconservative systems.

The non-Noether symmetry (3) of the nonconservative system need not be a “new” symmetry, and its generators $\xi_s$ and $\xi_0$ may be expressible in terms of the Lie point symmetries of the system. In fact, for a nonconservative dynamical system in which the Lie point symmetrical generators form a complete set of Lie point symmetries, any additional symmetry of the motion must necessarily be a function of these generators of the Lie point symmetries.

3. The Noether symmetries derived from a non-Noether symmetry

Eq. (13) gives non-Noether conserved quantities. However, it may reduce to Noether conserved quantities on the following condition.

Theorem 2. If the non-Noether symmetries possess non-Noether conserved quantity of the form
\[
\Phi = -X^{(1)}(f) - 2 \frac{\partial^2}{\partial \dot{q}_s \partial \dot{q}_s} \left[ M_{kl} \left( \dot{\xi}_s - \dot{\xi}_s \dot{\xi}_0 \right) Q_s \right],
\]
then the symmetry group preserves the action, the non-Noether symmetry leads to a Noether symmetry and possesses the Noether conserved quantity
\[
I = \xi_0 L + (\xi_s - \dot{q}_s \xi_0) \frac{\partial L}{\partial \dot{q}_s} + G_N = \text{const}.
\]

Proof. For any functions $\xi_0(q,t), \xi_s(q,t)$ one can easily verify the following equation by straightforward
calculation:

\[ X^{(1)}(\frac{\partial^2 L}{\partial q_k \partial \dot{q}_s}) = \frac{\partial^2 X^{(1)}(L)}{\partial q_k \partial \dot{q}_s} - \frac{\partial L}{\partial \dot{q}_k} \frac{\partial (A_{1s})}{\partial \dot{q}_s} - M_{ms} \frac{\partial^2 L}{\partial \dot{q}_m \partial \dot{q}_s} - A_{ls} \frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s}, \]  

where

\[ A_{lk} = \frac{\partial \xi_l}{\partial q_k} - q_l \frac{\partial \xi_0}{\partial q_k} - \dot{\xi}_0 \delta_{lk}. \]  

Noether theorem pointed out that if \( \xi(t, \dot{q}) \), \( \dot{\xi}(t, \dot{q}) \) determine a Noether symmetry of the nonconservative dynamical systems, there exists a function \( G_N(t, \dot{q}) \) satisfying [6,7]

\[ X^{(1)}(L) + \dot{\xi}_0 L + (\dot{\xi}_s - \dot{\xi}_0) Q_s = -\dot{G}_N. \]  

Since the right-hand side of Eq. (22) is linear in terms of the velocities, it leads to

\[ \frac{\partial^2 X^{(1)}(L)}{\partial q_k \partial \dot{q}_s} = \frac{\partial^2 (\dot{\xi}_0 L)}{\partial q_k \partial \dot{q}_s} - \frac{\partial^2}{\partial q_k \partial \dot{q}_l} \left[ (\dot{\xi}_s - \dot{\xi}_0) Q_s \right]. \]  

Substituting Eq. (22) into Eq. (20), one obtains

\[ X^{(1)}(\frac{\partial^2 L}{\partial q_k \partial \dot{q}_s}) = \dot{\xi}_0 \frac{\partial^2 L}{\partial q_k \partial \dot{q}_s} - B_{ls} \frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s} - B_{lk} \frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s} - \frac{\partial^2}{\partial q_k \partial \dot{q}_l} \left[ (\dot{\xi}_s - \dot{\xi}_0) Q_s \right], \]  

where

\[ B_{ls} = \frac{\partial \xi_l}{\partial q_s} - q_l \frac{\partial \xi_0}{\partial q_s}. \]  

Let \( M_{ms} \) be the cofactor of \( \frac{\partial^2 L}{\partial q_m \partial \dot{q}_s} \) in the matrix formed by these second derivatives; then the properties of determinants yield

\[ M_{ms} \frac{\partial^2 L}{\partial q_m \partial \dot{q}_l} = D \delta_{sl}, \]  

and

\[ M_{ms} \frac{\partial}{\partial \rho} \frac{\partial^2 L}{\partial q_m \partial \dot{q}_s} = \frac{\partial D}{\partial \rho}, \]  

where \( D \) is given by (10), and \( \rho \) may be one of \( q_s, \dot{q}_s \), or \( t \). Multiplying Eq. (24) by \( M_{ms} \), summing on repeated indices, and using Eqs. (26) and (27), one gets

\[ X^{(1)}(\ln D) = \dot{\xi}_0 N - 2B_{ll} \]  

Using Eqs. (28) and (13), one knows that conservation of \( \Phi \) can be proved if the symmetry group leaves the action invariant. In this case, one has the classical Noether invariant (19) on the nonconservative system.

Thus we see that a conserved quantity can be associated with any symmetry of a nonconservative dynamical system. □

Further, the authors may derive non-Noether symmetries of Lagrangian systems by those of nonconservative systems.

We should indicate that the present Letter may lead to the non-Noether symmetries of the Lagrangian systems when the nonpotential generalized force of the systems vanishes. The conserved quantity associated with a Lagrangian systems as [2]

\[ \Phi = 2 \left( \frac{\partial \xi_0}{\partial q_s} - \dot{q}_s \frac{\partial \xi_0}{\partial q_l} \right) - N \dot{\xi}_0 + X^{(1)}(\ln D), \]  

which is the same as the result in [2]. Furthermore, if the symmetry leaves action invariant and \( \Phi \) vanishes, a Noether symmetry which possesses a classical Noether conserved quantity is obtained.

We should point out that the non-Noether symmetries \( \xi(t, \dot{q}) \), \( \dot{\xi}(t, \dot{q}) \) form a complete set of the Lie point symmetries, and the aggregate of transformation generators constitutes a non-Noether symmetry for the Lagrangian systems. In fact, the Lie invariants form a complete set of the conserved quantity.

4. An example

Consider a dynamical system with Lagrangian \( L = \dot{q}^2/2 \), subjected to a nonconservative force \( Q' = \dot{q}^2 \), its equation of motion is

\[ \ddot{q}_s = \dot{q}^2. \]  

The determining equation of the non-Noether symmetry of Eq. (30) under the infinitesimal transformation \( \xi = \xi(t, \dot{q}) \), \( \dot{\xi}_0 = \dot{\xi}_0(t, \dot{q}) \) is
where Lie point symmetries (36a)–(36c) are ordinary, and Lie point symmetries (36d) and (36e) possess non-Noether conserved quantities. That indicates that the non-Noether symmetry (33) is a complete set of Lie point symmetries for the determining equation (31); non-Noether symmetrical conserved quantity (35) is also a complete set of the conserved quantities of Lie point symmetry corresponding the nonconservative system.

Substituting $Q' = q^2$ into Eq. (18), if the system possesses a non-Noether symmetrical conserved quantity in the form

$$\Phi = -X(1)(2q) - \frac{\partial^2}{\partial q^2}(\xi - q\xi_0)q^2,$$  \hspace{1cm} (37)

then the system will lead to a Noether symmetry. There exists a function $G_N(t, q)$ such that

$$X(1)\left(\frac{1}{2}q^2\right) + \frac{1}{2}\xi_0q^2 + (\xi - q\xi_0)q^2 = -G_N,$$  \hspace{1cm} (38)

and with the Noether conserved quantity

$$I = \frac{1}{2}\xi_0q^2 + (\xi - q\xi_0)q + G_N,$$  \hspace{1cm} (39)

where the generators $\xi$, $\xi_0$ can be associated with Noether symmetries of the system.

References