Bifurcations of smooth and nonsmooth traveling wave solutions in a generalized degasperis–procesi equation

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Abstract

In this paper, we employ the bifurcation theory of planar dynamical systems to study the smooth and nonsmooth traveling wave solutions of the generalized Degasperis-Procesi equation

\[ u_t - u_{xxt} + 4u^m_x u_x = 3u_x u_{xx} + uu_{xxx}. \]

The parameter condition under which peakons, compactons and periodic cusp wave solutions exist is given. The numerical simulation results show the consistence with the theoretical analysis at the same time.

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1. Introduction

Degasperis–Procesi equation \(^{[4]}\) and Camassa–Holm equation \(^{[1]}\) and their generalizations attracted many research attentions over years and have been studied successfully by many authors \(^{[3,5,7,8,11–13]}\) recently. Also some other generalized forms of these two equations have been investigated continuously. Degasperis et al. \(^{[5]}\) proved the integrability of DP equation by constructing a Lax pair of its and showed that the DP equation admits exact solutions in the form of a superposition of multi-peakons. Cooper and Shepard \(^{[3]}\) derived an approximate solitary wave solution to the CH equation by using some variational functions. Guo and Liu \(^{[7]}\) showed that the traveling wave solutions of DP equation and CH equation have the same topological bifurcation phase portraits, and gave the representations of their periodic cusp waves, single-solitons and peakons.

It has been well known that a lot of nonlinear dispersive equations such as Degasperis–Procesi equation and Camassa–Holm equation and their generalizations \(^{[7,11]}\) possess weak solution such as compactons, periodic cusp waves and peakons, etc.
In this paper, we study the generalized Degasperis–Procesi equation as follows:

\[ u_t - u_{xxt} + 4u^m u_x = 3u_x u_{xx} + uu_xxx, \]  

(1.1)

where \( m \in \mathbb{N} \). Clearly, when \( m = 1 \), (1.1) just is the well-known Degasperis–Procesi equation. Seldom has such equation as (1.1) taking an extra parameter \( m \) as the exponent of function been studied. Recently, Shen and Xu [11] did such a research. Unfortunately, the results are incomplete. Here, we are interested in a more complete study of the existence of traveling wave solutions of (1.1) in every parameter region of the parameter space. We show that Eq. (1.1) which admits the similar smooth and nonsmooth solutions as the Degasperis–Procesi equation, when \( m \) is an odd integer, but it is not the case when \( m \) is an even integer.

To investigate the traveling wave solution of Eq. (1.1), let \( u(x, t) = \phi(x - ct) = \phi(\xi) \), where \( c \) is the wave speed and \( \xi = x - ct \).

Substituting above traveling wave solutions into (1.1) allow us introduce the following ordinary differential equation:

\[ -c\phi' + c\phi''' + 4\phi^m \phi' = 3\phi \phi'' + \phi\phi''', \]  

(1.2)

where “’” denotes the derivative with respect to \( \xi \). Integrating (1.2) once with respect to \( \xi \) leads to

\[ -c\phi + (c - \phi)\phi'' + \frac{4}{m + 1} \phi^{m+1} - \phi'^2 = g, \]  

(1.3)

where \( g \) is an integral constant.

Eq. (1.3) is equivalent to the two-dimensional system as follows:

\[
\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -y^2 + \left(\frac{4}{m + 1}\right)\phi^{m+1} - c\phi - g
\]  

(1.4)

except for the solutions of (1.1) which satisfy \( \phi - c = 0 \) at some point \( \xi \).

Obviously, system (1.4) is a three-parameter planar dynamical system depending on the parameter group \( (c, m, g) \). Since the phase orbits defined by the vector fields of system (1.4) determine all traveling wave solutions of Eq. (1.1), we shall investigate the bifurcations of phase portraits of Eq. (1.4) in the phase plane \( (\phi, y) \) as the parameters \( c, m \) and \( g \) are changed. The bifurcation theory of dynamical systems (see [2,6,10]) plays an important role in our study.

We point out here that we are considering a physical model where only bounded traveling waves are meaningful. So we only pay attention to the bounded solutions of Eq. (1.4) in this paper.

We will show in this paper that the GDP Eq. (1.1) also has the similar topological bifurcation phase portraits as the DP equation and the CH equation when \( m \) is an odd number, and consequently it possesses the similar traveling wave solutions. The fact that besides the single-solitons, peakons, periodic cusp wave solutions, it has compacton solutions when \( m \) is an even number is also proved in this paper. We claim that the existence of a singular straight line for Eq. (1.4) is the original reason [9,14] why traveling waves lose their smoothness.

This paper is organized as follows. In Section 2, we obtain the bifurcation sets and phase portraits of the traveling wave solutions of the GDP equation. In Section 3, we consider the existence of the smooth solitary and periodic wave solutions of (1.4) and prove the existence of peakons, compactons and periodic cusp wave solutions. The numerical simulation results show the consistence with the theoretical analysis given at the same time.

2. Bifurcation sets and phase portraits of (1.4)

We will consider the phase portrait of Eq. (1.4) in this section. Let \( d\xi = (\phi - c)d\xi \), then system (1.4) is reduced to

\[
\frac{d\phi}{d\xi} = (\phi - c)y, \quad \frac{dy}{d\xi} = -y^2 + \frac{4}{m + 1} \phi^{m+1} - c\phi - g
\]  

(2.1)

which has the same first integral as system (1.4). Consequently, system (1.4) has the same topological phase portraits as system (2.1) except for the straight line \( \phi = c \). For the new system (2.1), \( \phi = c \) is an invariant straight-line solution.
Now we consider the singular points and their properties of system (2.1). Let
\[ f(\phi) = \frac{4}{m+1} \phi^{m+1} - c\phi - g = 0, \]
then and the first integral of system (2.1) can be expressed as follows:
\[ H(\phi, y) = y^2(\phi - c)^2 - \frac{8}{(m+1)(m+3)} \phi^{m+3} + \frac{8c}{(m+1)(m+2)} \phi^{m+2} + \frac{2c}{3} \phi^3 - (c^2 - g)^2 \phi - 2cg\phi = h. \] (2.3)

Note that for a fixed \( h \), (2.3) determines a set of invariant curves of (2.1) which contains two or three different branches of curves. As \( h \) is varied, (2.3) determined different families of orbits of (2.1) having different dynamical behaviors.

To investigate the equilibrium points of (2.1), we need to find all zeros of the function \( f(\phi) \). Let \( M(\phi_e, y_e) \) be the coefficient matrix of the linearized system of (2.1) at the equilibrium point \((\phi_e, y_e)\). Then
\[ M(\phi_e, y_e) = \begin{pmatrix} y_e & \phi_e - c \\ f'(\phi_e) & -2y_e \end{pmatrix}. \]

And at this equilibrium point, we have
\[ J(\phi_e, y_e) = \det M(\phi_e, y_e) = -2y_e^2 - (\phi_e - c)f'(\phi_e), \]
\[ p(\phi_e, y_e) = \text{trace}(M(\phi_e, y_e)) = -y_e^2 \leq 0. \] (2.4) (2.5)

By the theory of planar dynamical system (see [8,9]), for an equilibrium point of a planar integrable system, if \( J < 0 \), then this equilibrium point is a saddle point; It is a center point if \( J > 0 \) and \( p = 0 \); if \( J = 0 \) and the Poincaré index of the equilibrium point is 0, then it is a cusp. From above qualitative analysis, we can obtain the bifurcation curves and phase portraits under various parameter conditions as follows.

2.1. When \( m = 2n + 1 \), \( n \) is a natural number

In this section, we will investigate the bifurcation curves and phase portraits of (2.1) when \( m \) is an odd number. Let
\[ \phi_0 = \left( \frac{c}{4} \right)^{1/(2n+1)}, \]
\[ g_1(c) = -\frac{(2n+1)c}{2n+2} \left( \frac{c}{4} \right)^{1/(2n+1)}, \]
\[ g_2(c) = c^2 \left( \frac{4}{2n+2} - 1 \right). \] (2.6) (2.7) (2.8)

We can easily know that the function \( f(\phi) = 0 \) has two different real roots if the parameter \( g > g_1(c) \). There are two equilibrium points on the straight line \( \phi = c \) when \( g < g_2(c) \). Obviously, \( g_1(c) \leq g_2(c) \) and \( g_1(c) = g_2(c) \) if and only if \( c = 0 \) or \( c = \pm \left( \frac{1}{4} \right)^{1/2n} \) (only for \( n > 0 \)).

**Lemma 2.1.** Denote that \( y_\pm = \pm \sqrt{f(c)} \), \( \phi_- \leq \phi_0 \leq \phi_+ \), where \( \phi_- \), \( \phi_+ \) are the two real roots of \( f(\phi) = 0 \).

1. When \( g > g_2(c) \), system (2.1) has only two saddle points \( A_\pm(\phi_\pm, 0) \) lying on the each side of the straight line \( \phi = c \).
2. When \( g < g_1(c) \), system (2.1) has only two saddle points \( B_\pm(c, y_\pm) \) on the straight line \( \phi = c \).
3. When \( g_1(c) < g < g_2(c) \), system (2.1) has two saddle points \( B_\pm(c, y_\pm) \) on the straight line \( \phi = c \) and two critical points \( A_\pm(\phi_\pm, 0) \) on \( \phi \)-axis which both lie on the same side of the straight line \( \phi = c \). There are two different cases according to the different values of the wave speed \( c \).
Theorem 2.1. When \( 0 < c < \left( \frac{1}{4} \right)^{1/2n} \) or \( c < -\left( \frac{1}{4} \right)^{1/2n} \) for \( n > 0 \) (or when \( c < 0 \) for \( n = 0 \), \( A_\pm(\phi_\pm, 0) \) lie on the right side of the straight line \( \phi = c \). \( A_+ \) is a saddle point and \( A_- \) is a center point.

Case (b): When \(-\left( \frac{1}{4} \right)^{1/2n} < c < 0 \) or \( c > \left( \frac{1}{4} \right)^{1/2n} \) for \( n > 0 \) (or when \( c > 0 \) for \( n = 0 \), \( A_\pm(\phi_\pm, 0) \) lie on the left side of the straight line \( \phi = c \). \( A_- \) is a saddle point and \( A_+ \) is a center point.

(4) When \( g = g_1(c) \neq g_2(c) \), system (2.1) has two saddle points \( B_\pm(c, y_\pm) \) on the straight line \( \phi = c \) and a cusp \( (\phi_0, 0) \) on the \( \phi \)-axis.

(5) When \( g = g_2(c) \neq g_1(c) \), system (2.1) has a cusp \( (c, 0) \) and a saddle point which lie on the right side of the straight line \( \phi = c \) if \( 0 < c < \left( \frac{1}{4} \right)^{1/2n} \) or \( c < -\left( \frac{1}{4} \right)^{1/2n} \) and on the left side of the straight line \( \phi = c \) if \(-\left( \frac{1}{4} \right)^{1/2n} < c < 0 \) or \( c > \left( \frac{1}{4} \right)^{1/2n} \).

(6) When \( g = g_2(c) = g_1(c) \) i.e. when \( c = 0 \) or \( c = \pm \left( \frac{1}{4} \right)^{1/2n} \), system (2.1) has only a cusp.

To know the dynamical behavior and the orbits of system (2.1), we should investigate the values of the first integral \( H(\phi, y) \) at the critical points \( A_\pm \) and \( B_\pm \).

\[ H(A_\pm) = H(\phi_\pm, 0) = -\frac{2}{(n + 1)(n + 2)} \phi_\pm^{2n+4} + \frac{4c}{(n + 1)(2n + 3)} \phi_\pm^{2n+3} + \frac{2c}{3} \phi_\pm^3 - (c^2 - g) \phi_\pm^2 - 2cg \phi_\pm, \quad (2.9) \]

\[ H(B_\pm) = H(c, y_\pm) = -\frac{2}{(n + 1)(n + 2)(2n + 3)} c^{2n+4} - \frac{1}{3} c^4 - c^2 g. \quad (2.10) \]

As we known \( \phi_- \), \( \phi_+ \) are the two real roots of \( f(\phi) = 0 \). The values of \( \phi_- \), \( \phi_+ \) are different with diverse values of \( g \). If we regard as \( \phi_- \), \( \phi_+ \) as implicit functions of \( g \), we can easily get the following Lemma that describes the relationship between \( H(A_\pm) \) and \( H(B_\pm) \) under various parameter conditions by using the continuity of the function \( H(\phi_\pm, 0) \) about \( g \).

Lemma 2.2. Suppose that \( g_1(c) < g < g_2(c) \). Denote \( h_1^\pm = H(\phi_\pm, 0) = H(A_\pm), h_0 = H(B_\pm) = H(c, y) \).

(1) When \( 0 < c < \left( \frac{1}{4} \right)^{1/2n} \) or \( c < -\left( \frac{1}{4} \right)^{1/2n} \), there exists one and only one curve \( g = g_0^*(c) \) on \( (c, g) \)-plane on which \( h_1^+ = h_0 \) and \( h_1^- > h_0 \) when \( g > g_0^*(c) \) and \( h_1^- < h_0 \) when \( g < g_0^*(c) \).

(2) When \(-\left( \frac{1}{4} \right)^{1/2n} < c < 0 \) or \( c > \left( \frac{1}{4} \right)^{1/2n} \), there exists one and only one curve \( g = g_0^{**}(c) \) on \( (c, g) \)-plane on which \( h_1^+ = h_0 \) and \( h_1^- < h_0 \) when \( g < g_0^{**}(c) \) and \( h_1^- > h_0 \) when \( g > g_0^{**}(c) \).

According to the bifurcation theory, above analysis and the two Lemmas, we obtain the following theorem on the bifurcation curves and the phase portraits of system (2.1) when \( m \) is an odd number. Note that system (1.4) has the same topological phase portraits except for the straight line \( \phi = c \).

Theorem 2.1. When \( m = 2n + 1 \), for system (2.1), in \((c, g)\)-parameter plane, there exist four parametric bifurcation curves

\[ L_1 : g = g_1(c) = \frac{-(2n + 1)c^{(1/2n+1)}}{2n + 2} \left( \frac{c}{4} \right)^{(1/2n+1)}, \quad (2.11) \]

\[ L_2 : g = g_2(c) = c^2 \left( \frac{4}{2n + 2} c^{2n} - 1 \right). \quad (2.12) \]

\[ L_0^* : g = g_0^*(c) \quad \text{for } 0 < c < \left( \frac{1}{4} \right)^{1/2n}, \text{ or } c < -\left( \frac{1}{4} \right)^{1/2n}, \quad (2.13) \]

\[ L_0^{**} : g = g_0^{**}(c) \quad \text{for } -\left( \frac{1}{4} \right)^{1/2n} < c < 0 \text{ or } c > \left( \frac{1}{4} \right)^{1/2n}. \quad (2.14) \]
Fig. 1. Phase portrait of (2.1) for $m = 2n + 1$. (1) $(c, g) \in A_1$; (2) $(c, g) \in A_2$; (3) $(c, g) \in A_3$; (4) $(c, g) \in L_0^+$; (5) $(c, g) \in A_4$; (6) $(c, g) \in A_5$; (7) $(c, g) \in A_6$; (8) $(c, g) \in L_0^+$; (9) $(c, g) \in L_2$ and $-(\frac{1}{4})^{1/2n} < c < 0$ or $c > (\frac{1}{4})^{1/2n}$; (10) $(c, g) \in L_2 \cap L_1$; (11) $(c, g) \in L_2$ and $0 < c < (\frac{1}{4})^{1/2n}$ or $c < -(\frac{1}{4})^{1/2n}$; (12) $(c, g) \in L_1$ and $0 < c < (\frac{1}{4})^{1/2n}$ or $c < -(\frac{1}{4})^{1/2n}$; (13) $(c, g) \in L_1$ and $-(\frac{1}{4})^{1/2n} < c < 0$ or $c > (\frac{1}{4})^{1/2n}$.

These curves partition the $(c, g)$ parameter plane into six regions as follows:

- $A_1 : g > g_2(c)$, $A_2 : g < g_1(c)$,
- $A_3 : g_1(c) < g < g_0(c)$ and $0 < c < (\frac{1}{4})^{1/2n}$ or $c < -(\frac{1}{4})^{1/2n}$,
- $A_4 : g_0(c) < g < g_2(c)$ and $0 < c < (\frac{1}{4})^{1/2n}$ or $c < -(\frac{1}{4})^{1/2n}$,
- $A_5 : g_1(c) < g < g_0(c)$ and $-(\frac{1}{4})^{1/2n} < c < 0$ or $c > (\frac{1}{4})^{1/2n}$,
- $A_6 : g_0(c) < g < g_2(c)$ and $-(\frac{1}{4})^{1/2n} < c < 0$ or $c > (\frac{1}{4})^{1/2n}$.

The phase portraits in each region and on the bifurcation curves are shown, respectively, in Fig. 1.

2.2. When $m = 2n$, $n$ is a natural number

In this section, we will investigate the bifurcation curves and phase portraits of (2.1) when $m$ is an even number. It is very simple when $m = 0$, so we only consider the case when $m > 0$ in the following. When $c \leq 0$, $f'(\phi) = 4\phi^m -$
\[ \begin{align*}
c &= 4\phi^{2n} - c \geq 0 \quad \text{and } f(\phi) = (4/(2n+1))\phi^{2n+1} - c\phi - g = 0 \text{ has only one real root which we denoted by } \phi_0^*.
\end{align*} \]

Then at the critical point \((\phi_0^*, 0)\),
\[ J(\phi_0^*, 0) = -(\phi_0^* - c)f'(\phi_0^*). \] (2.15)

When \(c > 0\), \(f'(\phi) = 4\phi^{2n} - c = 0\) has two real roots. Write \(\phi_\pm^* = \pm(c/4)^{1/2n}\), and then \(f(\phi_\pm^*) = 0\) correspond to \(g = \pm g_1^*(c)\), respectively, where
\[ g_1^*(c) = -\frac{2n}{2n+1}c\left(\frac{c}{4}\right)^{1/2n}. \] (2.16)

Let
\[ g_2^*(c) = c^2\left(\frac{4}{2n+1}c^{2n-1} - 1\right), \] (2.17)

where \(c > 0\). Denote the only point at which the curves \(g = -g_1^*(c)\) and \(g_2^*(c)\) intersect by \((c^*, g^*)\). By similar analysis as in Section 2.1, we can obtain the following Lemmas.

**Lemma 2.3.** Let \(c^* = \left(\frac{1}{4}\right)^{2n-1}\).

*Case 1:* Suppose that \(c \leq 0\).

1. When \(g > g_2^*(c)\), system (2.1) has only one saddle point \((\phi_0^*, 0)\) lying on the right side of the straight line \(\phi = c\).
2. When \(g = g_2^*(c)\), system (2.1) has only one critical point \((c, 0)\).
3. When \(g < g_2^*(c)\), system (2.1) has two saddle points on the straight line \(\phi = c\) and one center point \((\phi_0^*, 0)\) which lies on the left side of the straight line \(\phi = c\).

*Case 2:* Suppose that \(c > 0\).

1. When \(g < g_1^*(c)\), system (2.1) has two saddle points on the straight line \(\phi = c\) and one center point \((\phi_0^*, 0)\) which lies on the left side of the straight line \(\phi = c\).
2. When \(0 < c < c^*\) and \(g > -g_1^*(c)\) or when \(c^* < c\) and \(g > g_2^*(c)\), system (2.1) has only a saddle point on the right side of the straight line \(\phi = c\); when \(c^* < c\) and \(g_2^*(c) > g > g_1^*(c)\), system (2.1) has two saddle points on the straight line \(\phi = c\) and a center point \((\phi_0^*, 0)\) on the left side of the straight line \(\phi = c\); when \(c^* < c\) and \(g = g_2^*(c)\), system (2.1) has a center point and a cusp on the left side of the straight line \(\phi = c\) and two saddle points on the straight line \(\phi = c\).
3. When \(-g_1^*(c) > g > g_1^*(c)\), the function \(f(\phi) = (4/(m+1))\phi^{m+1} - c\phi - g = 0\) has three different real roots \(\phi_1, \phi_2, \phi_3\) \((\phi_1 < \phi_2 < \phi_3)\). And then \(A_i(\phi_i, 0)\) \((i = 1, 2, 3)\) be three critical points of (2.1). There are three different cases.

(a) When \(0 < c < c^*\) and \(-g_1^*(c) > g > g_1^*(c)\), \(\phi_1 < \phi_2 < c < \phi_3\). Consequently, \(A_1\) is center point; \(A_2\) and \(A_3\) are two saddle points that lie on the each side of the straight line \(\phi = c\).

(b) When \(g = g_1^*(c)\) and \(0 < c < c^*, A_1\) is center point; \(A_2\) is a cusp on the straight line \(\phi = c\) and \(A_3\) is a saddle point which lies on the right side of the straight line \(\phi = c\). When \(g = g_2^*(c)\) and \(c^* < c < c^*, A_1\) is center point; \(A_2\) is a saddle point which lies on the left side of the straight line \(\phi = c\) and \(A_3\) is a cusp on the straight line \(\phi = c\).

(c) When \(0 < c < c^*\) and \(g_2^*(c) > g > g_1^*(c)\), system (2.1) has two saddle points on the straight line and \(\phi_1 < c < \phi_2 < \phi_3\). Consequently, \(A_1\) is center point which lie on the left side of the straight line \(\phi = c\); \(A_2\) is a center point and \(A_3\) is a saddle point and they all lie on the right side of the straight line \(\phi = c\).

When \(c^* < c\) and \(\min(-g_1^*(c), g_2^*(c)) > g > g_1^*(c)\), system (2.1) has two saddle points on the straight line and \(\phi_1 < \phi_2 < \phi_3 < c\). Consequently, \(A_1\) and \(A_2\) are two center points and \(A_3\) is a saddle point and they all lie on the right side of the straight line \(\phi = c\).

(4) When \(0 < c < c^*\) and \(g = -g_1^*(c)\), system (2.1) has no critical point on the straight line \(\phi = c\) and \(\phi_1 = \phi_2 < c < \phi_3\). Consequently, \(A_1\) is a cusp \(A_3\) is saddle point that lie on each side of the straight line \(\phi = c\). When \(c > c^*\) and \(g = -g_1^*(c)\),
system (2.1) has two saddle points on the straight line \( \phi = c \) and \( \phi_1 = \phi_2 < \phi_3 < c \). Consequently, \( A_1 \) is a cusp \( A_3 \) is center point and they all lie on the left side of the straight line \( \phi = c \).

(5) When \( 0 < c < c^{**} \) and \( g = g_1^*(c) \), system (2.1) has two saddle points on the straight line \( \phi = c \) and \( \phi_1 < \phi_2 = \phi_3 < c \). Consequently, \( A_1 \) is a center point and \( A_3 \) is a cusp. They lie on each side of the straight line \( \phi = c \). When \( c^{**} < c \) and \( g = g_1^*(c) \), system (2.1) has two saddle points on the straight line \( \phi = c \) and \( \phi_1 < \phi_2 = \phi_3 < c \). Consequently, \( A_1 \) is a center point and \( A_3 \) is a cusp. They all lie on the left side of the straight line \( \phi = c \). When \( c = c^{**} \) and \( g = g_1^*(c) \), system (2.1) has only two critical points on \( \phi \)-axis and \( \phi_1 < \phi_2 = \phi_3 = c \). Consequently, \( A_1 \) is a center point and \( A_3 \) is a cusp that lies on the straight line \( \phi = c \).

Lemma 2.4. Suppose \(-g_1^*(c) > g > g_1^*(c)\). Denote \( h_i = H(A_i), i = 1, 2, 3, h_0 = H(B_{\pm}) = H(c, y)\).

(1) When \( 0 < c < c^{**} \) and \( g_2^*(c) > g > g_1^*(c) \), there exists one and only one curve \( \hat{L}_0^* : g = \tilde{g}_0^*(c) \) on \((c, g)\)-plane that \( h_3 = h_0 \) when \( g = \tilde{g}_0^*(c) \); \( h_3 < h_0 \) when \( g < \tilde{g}_0^*(c) \) and \( h_3 > h_0 \) when \( g > \tilde{g}_0^*(c) \).

(2) When \( c^{**} < c \) and \( \min(-g_1^*(c), g_2^*(c)) > g > g_1^*(c) \), there exists one and only one curve \( \hat{L}_0^* : g = \hat{g}_0^*(c) \) on \((c, g)\)-plane that \( h_2 = h_0 \) when \( g = \hat{g}_0^*(c) \); \( h_2 > h_0 \) when \( g > \hat{g}_0^*(c) \) and \( h_2 < h_0 \) when \( g < \hat{g}_0^*(c) \).

According to the bifurcation theory and above analysis and the two Lemmas, we obtain the following theorem on the bifurcation curves of the phase portraits of system (2.1) when \( m \) is an even number.

Theorem 2.2. When \( m = 2n \), for system (2.1), in \((c, g)\)-parameter plane, there exist five parametric bifurcation curves:

\[
\hat{L}_1 : g = g_1^*(c) = -\frac{2n}{2n+1}c^{1/2n} \quad \text{for} \quad c > 0, \tag{2.18}
\]

\[
\hat{L}_2 : g = -g_1^*(c) = \frac{2n}{2n+1}c^{1/2n} \quad \text{for} \quad c > 0, \tag{2.19}
\]

\[
\hat{L}_3 : g = g_2^*(c) = c^2 \left( \frac{4}{2n+1}c^{2n-1} - 1 \right), \tag{2.20}
\]

\[
\hat{L}_0^* : g = \tilde{g}_0^*(c) \quad \text{for} \quad 0 < c < c^{**}, \tag{2.21}
\]

\[
\hat{L}_0^* : g = \hat{g}_0^*(c) \quad \text{for} \quad c^{**} < c. \tag{2.22}
\]

These curves partition the \((c, g)\) parameter plane into eight regions as follows:

\[ B_1 : g > g_2^*(c) \quad \text{for} \quad c = 0 \quad \text{or} \quad g > -g_1^*(c) \quad \text{for} \quad 0 < c < c^* \quad \text{or} \quad g > g_2^*(c) \quad \text{for} \quad c^* < c, \]

\[ B_2 : g < g_2^*(c) \quad \text{for} \quad c = 0 \quad \text{or} \quad g < g_1^*(c) \quad \text{for} \quad c > 0, \]

\[ B_3 : g_2^*(c) > g > -g_1^*(c) \quad \text{for} \quad c^* < c, \]

\[ B_4 : -g_1^*(c) > g > g_2^*(c) \quad \text{for} \quad 0 < c < c^*, \]

\[ B_5 : g_2^*(c) > g > \tilde{g}_0^*(c) \quad \text{for} \quad 0 < c < c^{**}, \]

\[ B_6 : \tilde{g}_0^*(c) > g > g_1^*(c) \quad \text{for} \quad 0 < c < c^{**}, \]

\[ B_7 : \min(-g_1^*(c), g_2^*(c)) > g > \tilde{g}_0^*(c) \quad \text{for} \quad c^{**} < c, \]

\[ B_8 : \tilde{g}_0^*(c) > g > g_1^*(c) \quad \text{for} \quad c^{**} < c. \]

The phase portraits in each region and on the bifurcation curves are shown in Fig. 2.

3. Dynamical behaviors and smooth and nonsmooth traveling wave solutions of (1.4)

In this section, we consider the existence of smooth and nonsmooth traveling wave solutions of (1.4). We first notice that system (1.4) has the same orbits as system (2.1) except for the straight line \( \phi = c \). The transformation of variables \( \tilde{\zeta} = (\phi - c)\zeta \) only derivates the difference of the parametric representations and the direction (when \( \phi - c < 0 \)) of orbits of systems (1.4) and (2.1) when \( \phi \neq c \). If the orbit of (2.1) has no intersection point with the straight line \( \phi = c \), then \( \phi' \) is well defined in (2.1). It follows that on \((\phi, y)\)-plane the profile defined by this orbit is smooth. If an orbit of (2.1) intersects with \( \phi = c \), then we know that the profile defined by this orbit is nonsmooth by similar analysis as in
Fig. 2. Phase portrait of (2.1) for $m = 2n$. (1) $(c, g) \in B_1$; (2) $(c, g) \in \tilde{L}_3(c < c^*)$; (3) $(c, g) \in \hat{L}_3(c^* < c < c^*)$; (4) $(c, g) \in B_7$; (5) $(c, g) \in B_5$; (6) $(c, g) \in \tilde{L}_2^*$; (7) $(c, g) \in B_6$; (8) $(c, g) \in B_5$; (9) $(c, g) \in \tilde{L}_2^*$; (10) $(c, g) \in B_1$; (11) $(c, g) \in \hat{L}_3(c \leq 0)$; (12) $(c, g) \in B_2$; (13) $(c, g) \in \tilde{L}_1(c > c^*)$; (14) $(c, g) \in B_3$; (15) $(c, g) \in \hat{L}_1$ for $0 < c < c^*$; (16) $(c, g) \in \hat{L}_1$ for $c^* < c$; (17) $(c, g) = (c^*, g^*)$; (18) $(c, g) = (c^*, g^*)$; (19) $(c, g) \in \hat{L}_2(c^* < c < c^*)$; (20) $(c, g) \in \hat{L}_2(0 < c < c^*)$. 
Theorem 3.1. Suppose that $m = 2n + 1$. Denote $h_{1}^{\pm} = H(\phi_{\pm}, 0)$, $h_{0} = H(c, y)$.

1. When $(c, g) \in A_{3}$, for $h \in (h_{1}^{-}, h_{1}^{+})$ in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). Corresponding to $h = h_{1}^{+}$, Eq. (1.4) has a smooth solitary wave solution of valley form.

2. When $(c, g) \in A_{4}$, for $h \in (h_{1}^{-}, h_{0})$ in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As $h$ increases from $h_{1}^{-}$ to $h_{0}$, the periodic traveling waves will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp waves, and the periods of the cusp waves tend to a constant finally (see Fig. 3).

3. When $(c, g) \in A_{5}$, for $h \in (h_{1}^{+}, h_{0})$ in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As $h$ increases from $h_{1}^{+}$ to $h_{0}$, the periodic traveling waves will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp waves, and tend to a peakon solution of valley form eventually, i.e. corresponding to $h = h_{0}$ in (2.3), (1.4) has a peakon solution of valley form (see Fig. 5).
Fig. 5. From smooth periodic traveling wave evolves to periodic cusp wave and tends to peakon wave as \( h \) increases from \( h^{-1} \) to \( h_{0} \), in the case \((c, g) \in L_{0}^{*} \) when \( m = 2n + 1 \). (1) \( h = 0.001542751101 \), (2) \( h = 0.0015692720267 \), (3) \( h = 0.001592734802 \) = \( h_{0} \) (\( m = 3, c = 0.2, g = -0.053045036705 \)).

(6) When \((c, g) \in L_{0}^{*} \), for \( h \in (h^{-1}, h_{0}) \) in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As \( h \) increases from \( h^{-1} \) to \( h_{0} \), the periodic traveling waves will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp waves, and tend to a peakon solution of peak form eventually, i.e. corresponding to \( h = h_{0} \) in (2.3), (1.4) has a peakon solution of peak form (see Fig. 6).

By above analysis and Lemma 2.4, we can easily get the following conclusions.

**Theorem 3.2.** Suppose that \( m = 2n \). Denote that \( H(q_{0}^{*}, 0) = h^{*} \).

(1) When \((c, g) \in B_{2} \) or \( B_{3} \), for \( h \in (h^{*}, h_{0}) \) in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As \( h \) increases from \( h^{*} \) to \( h_{0} \), the periodic traveling wave gradually loses its smoothness, and evolves from smooth periodic traveling wave to periodic cusp wave, and the periods of the cusp waves tend to a constant finally (similar to Fig. 4).

(2) When \((c, g) \in B_{4} \) or \((c, g) \in \hat{L}_{3} \) and, \( c^{*} < c < c^{**} \), for \( h \in (h_{1}, h_{2}) \) in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). Corresponding to \( h = h_{2} \), Eq. (1.4) has a smooth solitary wave solution of valley form.

(3) When \((c, g) \in B_{5} \), for \( h \in (h_{1}, h_{0}) \) or \( h \in (h_{2}, h_{0}) \) in (2.3), (1.4) has two families of uncountably infinite many smooth periodic traveling wave solutions. As \( h \) increases from \( h_{1} \) (or \( h_{2} \)) to \( h_{0} \), the periodic traveling waves will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp waves, and the periods of the two families of cusp waves tend to a constant, respectively (similar to Figs. 4 and 5).

(4) When \((c, g) \in \hat{L}_{0}^{*} \), for \( h \in (h_{1}, h_{0}) \) or \( h \in (h_{2}, h_{0}) \) in (2.3), (1.4) has two families of uncountably infinite many smooth periodic traveling wave solutions. As \( h \) increases from \( h_{1} \) (or \( h_{2} \)) to \( h_{0} \), the periodic traveling waves will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp waves, and the periods of one family of the cusp waves tend to a constant, but the periods of another family of the cusp waves...
Fig. 7. From smooth periodic traveling wave evolves to smooth solitary to smooth periodic traveling wave and tends to periodic cusp wave as $h$ increases from $h_1$ to $h_2$ to $h_0$, in the case $(c, g) \in L^*_1$ and $c^{**} < c$ when $m = 2n$. (1) $h = -0.40334157 > h_1$, (2) $h = 0.189225015$, (3) $h = 0.189709314 = h_2$, (4) $h = 0.253042421$, (5) $h = 0.271552386 = h_0$ ($m = 2, c = 1.2, g = -0.438178046$).

approach to $+\infty$, i.e. corresponding to $h = h_0$ in (2.3), (1.4) has a periodic cusp wave solution and a peakon solution of valley form (similar to Figs. 4 and 5).

(5) When $(c, g) \in B_6$, for $h \in (h_1, h_0)$ in (2.3), there is a family of uncountably infinite many smooth periodic traveling wave solutions of (1.4). As $h$ increases from $h_1$ to $h_0$, the periodic traveling wave will gradually lose their smoothness, and evolves from smooth periodic traveling waves to periodic cusp waves, and the periods of the cusp waves tend to a constant. For $h \in (h_3, h_4)$ in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). Corresponding to $h = h_4$, Eq. (1.4) has a smooth solitary wave solution of valley form (similar to Fig. 4).

(6) When $(c, g) \in B_7$, for $h \in (h_1, h_2)$ in (2.3), there is a family of uncountably infinite many smooth periodic traveling wave solutions of (1.4). Corresponding to $h = h_2$, Eq. (1.4) has a smooth solitary wave solution of valley form. For $h \in (h_3, h_0)$ in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As $h$ increases from $h_3$ to $h_0$, the periodic traveling wave gradually loses its smoothness, and evolves from smooth periodic traveling wave to periodic cusp wave, and the periods of the cusp waves tend to a constant finally (similar to Fig. 4).

(7) When $(c, g) \in \hat{L}^*_0$, for $h \in (h_1, h_0)$ in (2.3), there is a family of uncountably infinite many smooth periodic traveling wave solutions of (1.4). For $h \in (h_3, h_0)$ in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As $h$ increases from $h_3$ to $h_0$, the periodic traveling wave gradually loses its smoothness, and evolves from smooth periodic traveling wave to periodic cusp wave, and the period of the cusp wave approach to $-\infty$. Corresponding to $h = h_0$ in (2.3), Eq. (1.4) has a periodic cusp wave solution and a peakon solution of peak form and a smooth solitary wave solution of valley form (similar to Fig. 6).

(8) When $(c, g) \in B_8$, for $h \in (h_1, h_2)$ or $(h_3, h_2)$ in (2.3), there are two families of uncountably infinite many smooth periodic traveling wave solutions of (1.4). Corresponding to $h = h_2$, Eq. (1.4) has a smooth solitary wave solution of valley form and one of peak form. For $h \in (h_2, h_0)$ in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As $h$ increases from $h_2$ to $h_0$, the periodic traveling wave gradually loses its smoothness, and evolves from smooth periodic traveling wave to periodic cusp wave, and the periods of the cusp waves tend to a constant finally (similar to Fig 4).
(9) When \((c, g) \in \hat{L}_1\) and \(0 < c < c^{**}\), for \(h \in (h_1, h_0)\) in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As \(h\) increases from \(h_1\) to \(h_0\), the periodic traveling wave gradually loses its smoothness, and evolves from smooth periodic traveling wave to periodic cusp wave, and the periods of the cusp waves tend to a constant finally (similar to Fig. 4). When \((c, g) \in \hat{L}_1\) and \(c^{**} < c\), for \(h \in (h_1, h_2)\) in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4); corresponding to \(h = h_2\), Eq. (1.4) has a smooth solitary wave solution of valley form; for \(h \in (h_2, h_0)\) in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). As \(h\) increases from \(h_2\) to \(h_0\), the periodic traveling wave gradually loses its smoothness, and evolves from smooth periodic traveling wave to periodic cusp wave, and the periods of the cusp waves tend to a constant finally (see Fig. 7).

(10) When \((c, g) \in \hat{L}_3\) and \(0 < c < c^{*}\), for \(h \in (h_1, h_0)\) in (2.3), there are uncountably infinite many smooth periodic traveling wave solutions of (1.4). Corresponding to \(h = h_0\), Eq. (1.4) has a compaton (see Fig. 8).

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